HALE SCHOOL

Semester Two Examination, 2019

Question/Answer booklet

Year 11

MATHEMATICS METHODS
UNITS 1 AND 2

Section Two:
Calculator-assumed

SOLUTIONS

Time allowed for this section
Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	50	35
Section Two: Calculator-assumed	13	13	100	90	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (90 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

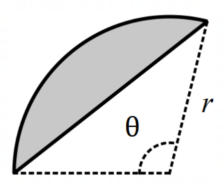
Working time: 100 minutes.

Question 9 **(6 marks)**

(a) Convert $96°$ to an exact radian measure. (1 mark)

Solution
$96 \times \frac{\pi}{180} = \frac{8\pi}{15}$
Specific behaviours
✓ correct value

(b) A segment of a circle of radius 33 cm is shown below, where $\theta = 96°$.



(i) Determine the area of the segment. (2 marks)

Solution
$A = \frac{1}{2}(33)^2\left(\frac{8\pi}{15} - \sin\frac{8\pi}{15}\right)$ $\approx 370.80 \text{ cm}^2$
Specific behaviours
✓ indicates correct use of formula ✓ correct area

(ii) Determine the perimeter of the segment. (3 marks)

Solution
Arc length is L and chord length is C. $L = 33 \times \frac{8\pi}{15} \approx 55.29$ $C^2 = 33^2 + 33^2 - 2(33)(33)\cos 96°$ OR $C = 2 \times 33 \times \sin\left(0.5 \times \frac{8\pi}{15}\right)$ $C \approx 49.05$ $P \approx 55.29 + 49.05$ $\approx 104.34 \text{ cm}$
Specific behaviours
✓ arc length ✓ chord length ✓ correct perimeter

Question 10 **(3 marks)**

Find the size of angle Q in triangle PQR given that angle P measures 56°, PR = 13.5 metres and QR = 16.8 metres.

Solution
$$\frac{13.5}{\sin Q} = \frac{16.8}{\sin 56}$$ $$Q \approx 41.77° \; or \; 138.23°$$ But need to discount $138.23°$ because otherwise angles in triangle add up to more than $180°$
Specific behaviours
✓ indicates correct use of formula ✓ first solution ✓ finds second solution but discounts it in context

Question 11 **(7 marks)**

From a random survey of telephone usage in 261 households it was found that 155 households had access to both mobiles and landlines, 54 households had no access to a mobile and 145 more households had landlines than did not.

(a) Complete the missing entries in the table below. (3 marks)

	Mobile	No mobile	Total
Landline	155	**48**	**203**
No landline	**52**	**6**	**58**
Total	**207**	**54**	261

Solution
See table
$x + (x + 145) = 261 \Rightarrow x = 58$
Specific behaviours
✓ totals column
✓ totals row
✓ rest of table

(b) If one household is randomly selected from those surveyed, determine the probability that

(i) it had access to a mobile phone. (1 mark)

Solution
$P(M) = 207 \div 261 \approx 0.793$
Specific behaviours
✓ correct probability

(iii) it had access to a mobile given that it no access to a landline. (1 mark)

Solution
$P(M\|\bar{L}) = 52 \div 58 \approx 0.897$
Specific behaviours
✓ correct probability

(c) Use your answers from part (b) to comment on the possible independence of households having access to a landline and households having access to a mobile phone. (2 marks)

Solution
No indication that the events are independent as $P(M) \neq P(M\|\bar{L})$ - would expect these probabilities to be closer if independent.
Specific behaviours
✓ justifies by comparing prob for independent events
✓ draws valid conclusion

Question 12 **(10 marks)**

When a manufacturer makes x litres of a chemical using process B, the cost in dollars per litre $C(x)$ varies according to the rule

$$C(x) = \frac{300}{x+10}, \qquad 5 \le x \le 65.$$

(a) Determine

(i) the cost per litre when 38 L is made. (1 mark)

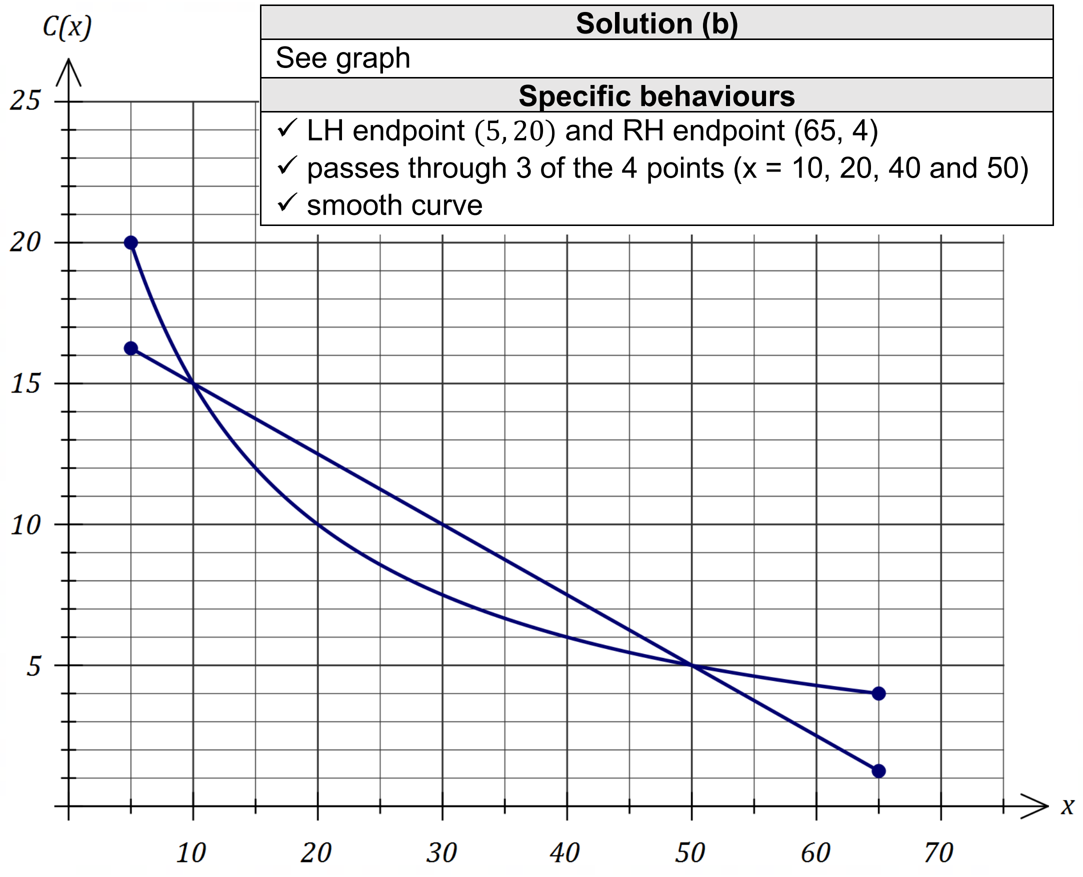
Solution
$C(38) = 6.25$ \$/L
Specific behaviours
✓ correct cost per litre

(ii) the total cost of making 14 L of the chemical. (2 marks)

Solution
$C(14) = 12.5$ $T = 12.5 \times 14 = \$175$
Specific behaviours
✓ cost per litre ✓ total cost

(b) Graph the cost per litre over the given domain on the axes below. (3 marks)

Solution (b)
See graph
Specific behaviours
✓ LH endpoint $(5, 20)$ and RH endpoint (65, 4) ✓ passes through 3 of the 4 points (x = 10, 20, 40 and 50) ✓ smooth curve



(c) State the range of $C(x)$. (1 mark)

Solution
$4 \leq C(x) \leq 20$
Specific behaviours
✓ correct range

When the manufacturer uses process D, the cost in dollars per litre $K(x)$ is modelled by

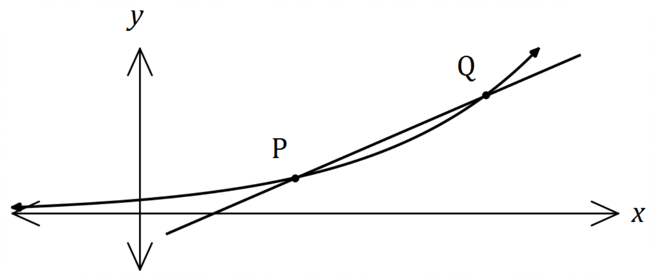
$$K(x) = \frac{35}{2} - \frac{x}{4}, \qquad 5 \leq x \leq 65.$$

(d) Add this function to the graph and hence determine the production quantities for which process B is cheaper than process D. (3 marks)

Solution
See graph for line. Process B is cheaper than Z for $10 < x < 50$ litres.
Specific behaviours
✓ ruled line through $(10, 15)$ and $(50, 5)$ ✓ correct endpoints ✓ states the correct range

Question 13 **(5 marks)**

The graph of $y = f(x)$ is shown below, where $f(x) = 4^x$, together with the secant to the curve through the points P and Q.



P has coordinates $(1, 4)$ and Q has coordinates $\big(1 + h, f(1 + h)\big)$ where $0 < h \leq 1$.

(a) Show that the gradient of $PQ = 12$ when $h = 1$. (1 mark)

Solution
$\frac{f(2) - f(1)}{1} = \frac{16 - 4}{1} = 12$
Specific behaviours
✓ shows substitution correctly

(b) Complete the second column in the table below, rounding values to 4 decimal places. (2 marks)

h	$\frac{f(1+h) - f(1)}{h}$
1	**12**
0.1	**5.9479**
0.01	**5.5838**
0.001	**5.5490**

Solution
See table
Specific behaviours
✓ evaluates at least one correctly ✓ all values are correct and rounded to 4 dp

(c) Determine an estimate, correct to 3 decimal places, for the value that $\frac{f(1+h) - f(1)}{h}$ approaches as h becomes closer and closer to 0 and state what this value represents. (2 marks)

Solution
Value approaches 5.545 (3 dp). Value is gradient of curve at P.
Specific behaviours
✓ correct value rounded to 3dp ✓ states value approaches gradient at point (or algebraic interpretation)

Question 14 **(5 marks)**

A geometric sequence has a second term of -8.4 and a sum to infinity of 15.

Determine the sum of the first 4 terms of the sequence.

Solution

$$ar = -8.4, \qquad \frac{a}{1-r} = 15$$

$$\begin{cases} a \times r = -8.4 \\ \dfrac{a}{1-r} = 15 \end{cases}\Bigg|_{a,\,r}$$

$$\left\{\left\{a=-6,\ r=\frac{7}{5}\right\},\ \left\{a=21,\ r=-\frac{2}{5}\right\}\right\}$$

Solving simultaneously gives $a = 21, r = -0.4$

(ignore $r = 1.4$ since $|r| < 1$ for sum to infinity)

$$S_4 = \frac{21(1-(-0.4)^4)}{1-(-0.4)}$$
$$= \frac{1827}{125} = 14.616$$

Specific behaviours

- ✓ equation using T_2
- ✓ equation using S_∞
- ✓ solves for a and r
- ✓ discards invalid solution and states why
- ✓ calculates S_4

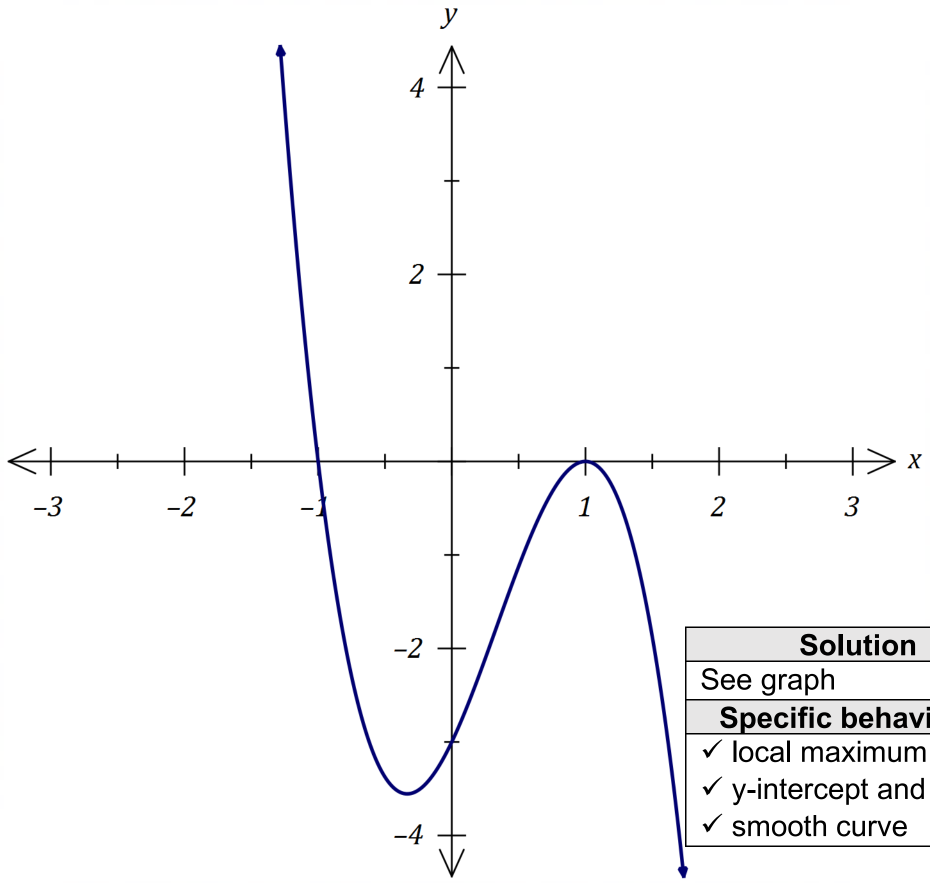
Question 15 **(12 marks)**

The function f is defined by $f(x) = -3x^3 + px^2 + qx + r$, where $p,\ q$ and r are constants.

The graph of $y = f(x)$ has the following features:

- passes through $(0, -3)$ and $(-1, 0)$
- has a local maximum at $(1, 0)$

(a) Sketch the graph of $y = f(x)$ on the axes below. (3 marks)



Solution
See graph
Specific behaviours
✓ local maximum
✓ y-intercept and root
✓ smooth curve

(b) Determine the value of p, the value of q and the value of r. (3 marks)

Solution 1
$f(x) = -3(x+1)(x-1)^2$ $= -3x^3 + 3x^2 + 3x - 3$ $p = 3, \quad q = 3, \quad r = -3$
Specific behaviours
✓ writes in factored form
✓ expands
✓ states all three values

Solution 2
$f(0) = -3$ $r = -3$ f(1) = 0 so p + q = 6   f(-1) = 0 so p – q = 0   f'(1) = 0 so 2p+q = 9 $p = 3, \quad q = 3, \quad r = -3$
Specific behaviours
✓ finds r = - 3
✓ finds a connection between p, q
✓ states all three values

(c) Use a calculus method to determine the exact coordinates of the local minimum of the graph of $y = f(x)$. Justifying the nature of turning point is not necessary. (3 marks)

Solution
$f'(x) = -9x^2 + 6x + 3$ $f'(x) = 0 \Rightarrow x = -\frac{1}{3}, 1$ $f\left(-\frac{1}{3}\right) = -\frac{32}{9} \quad (-3.\overline{5})$ Local minimum at $\left(-\frac{1}{3}, -\frac{32}{9}\right)$
Specific behaviours
✓ shows $f'(x)$ ✓ shows $f'(x) = 0$ and solutions   ✓ correct coordinates

(d) Determine the coordinates of the point where the tangent to $y = f(x)$ at $(0, -3)$ intersects the curve $y = f(x)$, other than at the point of tangency. (3 marks)

Solution
$f'(0) = 3$ Tangent: $y = 3x - 3$ $-3x^3 + 3x^2 + 3x - 3 = 3x - 3$ $x = 0, x = 1$ Intersects at $(1, 0)$
Specific behaviours
✓ equation of tangent   ✓ equates tangent to curve and solves   ✓ correct coordinates

Question 16 **(7 marks)**

The amount of water in a tank, W litres, varies with time t, in minutes, and can be modelled by the equation $W = 200 - 185(1.2)^{-t}$, $t \geq 0$.

(a) Determine amount of water in the tank

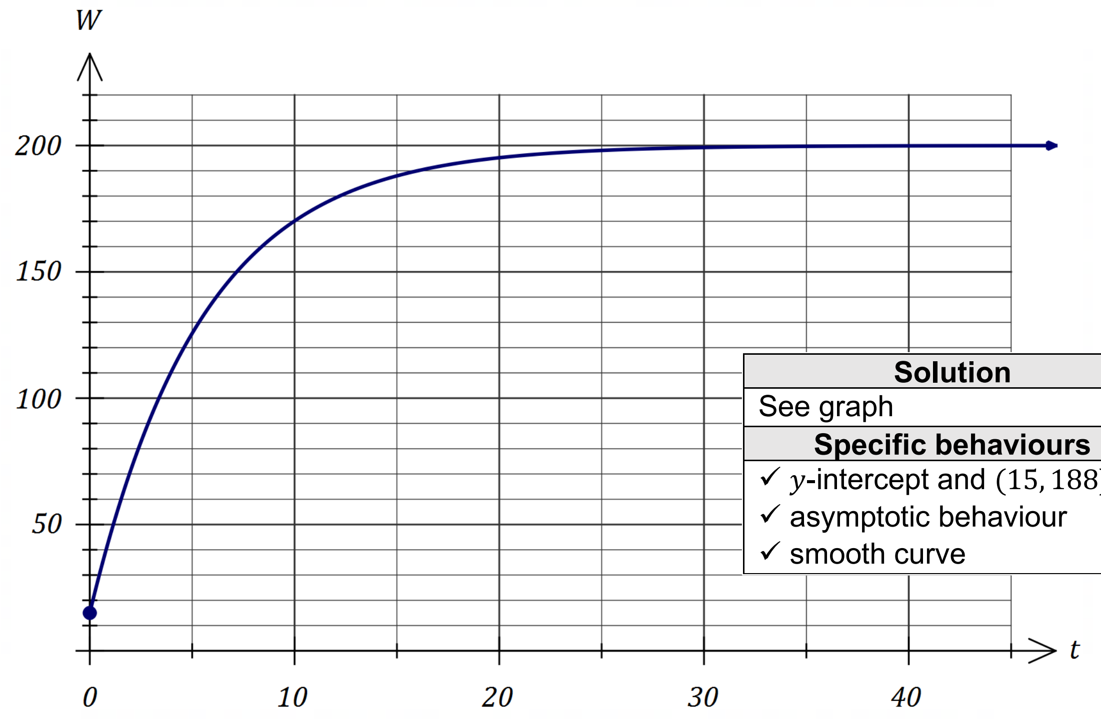
(i) initially. (1 mark)

Solution
$W(0) = 15$ L
Specific behaviours
✓ correct value

(ii) after 15 minutes. (1 mark)

Solution
$W(15) = 187.99$ L
Specific behaviours
✓ correct value

(b) Graph W against t for $0 \leq t \leq 45$ on the axes below. (3 marks)



Solution
See graph
Specific behaviours
✓ y-intercept and $(15, 188)$
✓ asymptotic behaviour
✓ smooth curve

(c) Over time, the amount of water in the tank approaches v litres. State the value of v and the value of t at which the amount of water in the tank reaches 99% of v. (2 marks)

Solution
$v = 200$ L $W = 0.99(200) \Rightarrow t = 24.8$ minutes
Specific behaviours
✓ correct value of v ✓ correct time

Question 17 **(7 marks)**

When a patient takes a painkilling drug D, the probability that they experience some side effects is known to be 0.2.

(a) A doctor prescribes drug D to two unrelated patients. Determine the probability that

(i) neither patient experiences some side effects. (1 mark)

Solution
$P = (0.8)^2 = 0.64$
Specific behaviours
✓ correct probability

(ii) one patient experiences some side effects and the other does not. (2 marks)

Solution
$P = 0.2 \times 0.8 \times 2$ $= 0.32$
Specific behaviours
✓ calculates $p(1-p)$ ✓ doubles to obtain final probability

Other painkilling drugs are available. Of those who take drug D, 75% of patients who suffer some side effects will switch to another drug whereas no patient who has no side effects will switch.

(b) The doctor prescribes drug D to a patient. Determine the probability that the patient does not switch to another drug. (2 marks)

Solution
$P = 0.8 + 0.2 \times 0.25$ $= 0.8 + 0.05$ $= 0.85$
Specific behaviours
✓ probability of side effect and does not switch ✓ correct probability

(c) The doctor prescribes drug D to three unrelated patients. Determine the probability that at least one of these patients switch to another drug. (2 marks)

Solution
$P(\text{none}) = 0.85^3$ ≈ 0.6141 $P = 1 - 0.6141$ ≈ 0.3859
Specific behaviours
✓ probability none switch ✓ correct probability

Question 18 **(8 marks)**

Two events A and B are such that $P(A \cap \bar{B}) = x, P(A) = 0.2$ and $P(\bar{A} \cap B) = 0.6$.

(a) Determine $P(A \cap B)$ when $x = 0.12$. (2 marks)

Solution
$P(A \cap B) = P(A) - P(A \cap \bar{B})$ $= 0.2 - 0.12$ $= 0.08$
Specific behaviours
✓ use of Venn diagram or other method ✓ correct probability

(b) Determine an expression for $P(A \cap B)$ in terms of x. (1 mark)

Solution
$P(A \cap B) = 0.2 - x$
Specific behaviours
✓ correct expression

(c) Determine the value of x when

(i) A and B are mutually exclusive. (1 mark)

Solution
$P(A \cap B) = 0.2 - x = 0 \Rightarrow x = 0.2$
Specific behaviours
✓ correct value

(ii) A and B are independent. (2 marks)

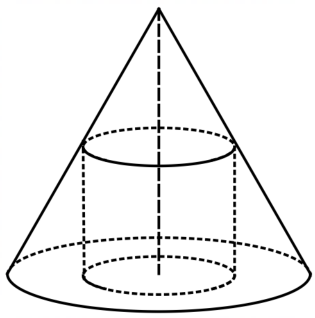
Solution
$P(A \cap B) = P(A) \times P(B)$ $0.2 - x = 0.2 \times (0.6 + 0.2 - x)$ $x = 0.05$
Specific behaviours
✓ uses rule for independence ✓ correct value

(iii) $P(A|B) = 0.04$. (2 marks)

Solution
$P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ $0.04 = \dfrac{0.2 - x}{0.6 + 0.2 - x}$ $x = 0.175$
Specific behaviours
✓ uses conditional probability rule ✓ correct value

Question 19 **(7 marks)**

A right circular cone of base radius 8 cm and height 24 cm stands on a horizontal surface. A cylinder of radius x cm and volume V cm^3 stands inside the cone with its axis coincident with that of the cone and such that the cylinder touches the curved surface of the cone as shown.



(a) Show that $V = 24\pi x^2 - 3\pi x^3$. (3 marks)

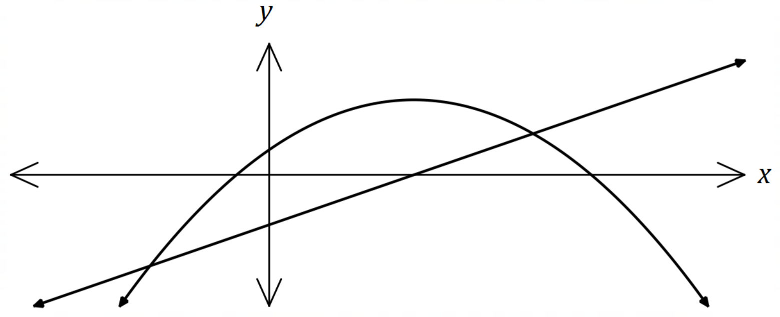
Solution
From similar triangles $$\frac{h}{8-x} = \frac{24}{8} \Rightarrow h = 24 - 3x$$ Hence $$V = \pi r^2 h$$ $$V = \pi x^2(24 - 3x)$$ $$= 24\pi x^2 - 3\pi x^3$$
Specific behaviours
✓ relation between two variables using similar triangles ✓ uses relationship to express h in terms of x ✓ substitutes into cylinder volume formula

(b) Given that x can vary, use a calculus method to determine the value of x for which V is a maximum. (4 marks)

Solution
$$\frac{dV}{dx} = 48\pi x - 9\pi x^2$$ $$\frac{dV}{dx} = 0 \text{ when } x = 0, x = \frac{16}{3}$$ $V''\left(\frac{16}{3}\right) < 0$ therefore maximum
Specific behaviours
✓ derivative ✓ equates derivative to 0 ✓ solves for x ✓ sign test or 2nd derivative test to prove max

Question 20 **(6 marks)**

The graphs of $y = f(x)$ and $y = g(x)$ are shown below where $f(x) = 1 + 4x - 2x^2$ and $g(x) = 2x + k$.



Determine the value(s) of the constant k so that the equation $f(x) = g(x)$ has

(a) one solution. (5 marks)

Solution
$1 + 4x - 2x^2 = 2x + k$ $0 = 2x^2 - 2x + k - 1$ $b^2 - 4ac = 0$ $(-2)^2 - 4(2)(k-1) = 0$ $k = 1.5$
Specific behaviours
✓ equate the lines ✓ rearrange to give 0 ✓ recognise that discriminant must equal 0 ✓ substitute values ✓ solves correctly

Alternative Solution
g must be a tangent to f: $f'(x) = 4 - 4x$ $= 2 \text{ when } x = \frac{1}{2}$ y-coordinate of point of tangency: $f\left(\frac{1}{2}\right) = 1 + 4\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^2 = \frac{5}{2}$ Equation of tangent: $y - \frac{5}{2} = 2\left(x - \frac{1}{2}\right)$ $y = 2x + \frac{3}{2}$ Hence $k = \frac{3}{2} = 1.5$
Specific behaviours
✓ indicates tangent required ✓ determines x-coordinate of point of tangency ✓ determines y-coordinate of point of tangency ✓ equation of tangent ✓ states correct value of k

(b) no solutions. (1 mark)

Solution
$k > 1.5$
Specific behaviours
✓ correct inequality

Question 21 **(7 marks)**

A fair four-sided dice numbered $1, 2, 3$ and 4 is thrown n times until it lands on a 4.

(a) Show that the probability that $n = 2$ is $\frac{3}{16}$. (1 mark)

Solution
$P(n = 3) = \frac{3}{4} \times \frac{1}{4} = \frac{3}{16}$
Specific behaviours
✓ shows product of two fractions

(b) Determine the probability that $n = 5$. (1 mark)

Solution
$P(n = 5) = \left(\frac{3}{4}\right)^4 \times \frac{1}{4} = \frac{81}{1024} \approx 0.0791$
Specific behaviours
✓ correct probability

(c) The probabilities form a geometric sequence. Write an expression in terms of n for the probability that the first 4 is thrown on the n^{th} throw. (1 mark)

Solution
$P = \frac{1}{4}\left(\frac{3}{4}\right)^{n-1}$
Specific behaviours
✓ correct expression

(d) Determine the probability that the first 4 is thrown in 7 or less attempts. (2 marks)

Solution
$S_7 = \dfrac{\frac{1}{4}\left(1 - \left(\frac{3}{4}\right)^7\right)}{1 - \frac{3}{4}} \approx 0.8665$
Specific behaviours
✓ indicates use of sum formula
✓ correct probability

Alternative Solution
$a_{n+1} = a_n \times \frac{3}{4}$
$a_1 = 0.25$
$Sum\ of\ 1st\ 7\ terms \approx 0.8665$
Specific behaviours
✓ states both parts of recursive equation
✓ correct probability

(e) The probability that the first 4 is thrown in k or less attempts must be at least 99.5%. Determine the least value of integer k. (2 marks)

Solution
$0.995 = \dfrac{\frac{1}{4}\left(1 - \left(\frac{3}{4}\right)^n\right)}{1 - \frac{3}{4}} \Rightarrow n = 18.4$
$k = 19$
Specific behaviours
✓ solves for n
✓ correct value of k

Alternative Solution
$S_{18} = 0.9944$
$S_{19} = 0.9958$
$k = 19$
Specific behaviours
✓ states values of surrounding sums
✓ correct value of k

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________